

A TRIDENT SCHOLAR PROJECT REPORT

NO. 239

NUMERICAL MODELING OF SLOPEWATER CIRCULATION



UNITED STATES NAVAL ACADEMY
ANNAPOLIS, MARYLAND

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REPORT DOCUMENTATION PAGE

Form Approved
OMB No. 0704-0188

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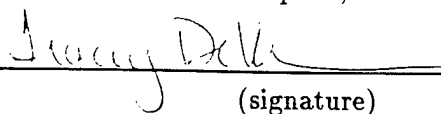
1. AGENCY USE ONLY (Leave Blank)		2. REPORT DATE 1996		3. REPORT TYPE AND DATES COVERED	
4. TITLE AND SUBTITLE Numerical modeling of slopewater circulation				5. FUNDING NUMBERS	
6. AUTHOR(S) Delk, Tracey					
7. PERFORMING ORGANIZATION NAME(S) AND ADDRESS(ES)				8. PERFORMING ORGANIZATION REPORT NUMBER	
9. SPONSORING/MONITORING AGENCY NAME(S) AND ADDRESS(ES) United States Naval Academy Annapolis, MD 21402				10. SPONSORING/MONITORING AGENCY REPORT NUMBER USNA Trident Scholar report; no. 239 (1996)	
11. SUPPLEMENTARY NOTES Accepted by the U.S.N.A. Trident Scholar Committee					
12a. DISTRIBUTION/AVAILABILITY STATEMENT Approved for public release; distribution unlimited.				12b. DISTRIBUTION CODE UL	
13. ABSTRACT (Maximum 200 words) The area between the continental shelf and the Gulf Stream is known as the slopewater region. In the past fifty years several experiments and studies of this region have taken place with the Mid-Atlantic Continental Slope and Rise (MASAR) experiment being one of the most recent. Csanady and Hamilton (1988) compiled all the known information and data from the slopewater region and developed a simple dynamic model of the flow. Based on this model's transport stream function and Stommel's Gulf Stream model, finite centered differencing was used to develop a numerical scheme of slopewater circulation. The model was first developed using Stommel's parameter for circulation within the North Atlantic Gyre. Stommel's model was used as the basis for the new scheme in order to calibrate the model with his exact solution of the stream function for the North Atlantic Gyre. Once verified, Stommel's parameters were replaced by Csanady and Hamilton's values for slopewater. This is a report on the development of a new numerical model. It is also a comparison of the new scheme to both Csanady and Hamilton's model and an observational schematic for the region from the MASAR experiment.					
14. SUBJECT TERMS Circulation; Slopewater; Modeling; Physical oceanography				15. NUMBER OF PAGES 44	
				16. PRICE CODE	
17. SECURITY CLASSIFICATION OF REPORT UNCLASSIFIED	18. SECURITY CLASSIFICATION OF THIS PAGE UNCLASSIFIED	19. SECURITY CLASSIFICATION OF ABSTRACT UNCLASSIFIED	20. LIMITATION OF ABSTRACT UL		

U.S.N.A. —Trident Scholar project report; no. 239 (1996)

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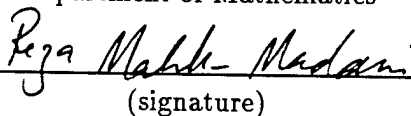
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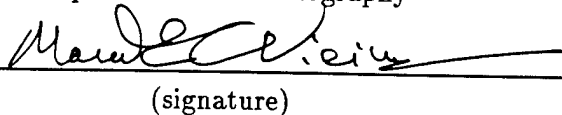
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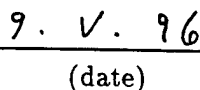
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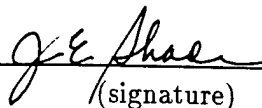
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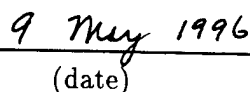

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Abstract

The area between the continental shelf and the Gulf Stream is known as the slopewater region. In the past fifty years several experiments and studies of this region have taken place with the Mid-Atlantic Continental Slope and Rise (MASAR) experiment being one of the most recent. Csanady and Hamilton (1988) compiled all the known information and data from the slopewater region and developed a simple dynamical model of the flow. Based on this model's transport stream function and Stommel's Gulf Stream model, finite centered differencing was used to develop a numerical scheme of slopewater circulation.

The model was first developed using Stommel's parameters for circulation within the North Atlantic Gyre. Stommel's model was used as the basis for the new scheme in order to calibrate the model with his exact solution of the streamfunction for the North Atlantic Gyre. Once verified, Stommel's parameters were replaced by Csanady and Hamilton's values for slopewater. This is a report on the development of a new numerical model. It is also a comparison of the new scheme to both Csanady and Hamilton's model and an observational schematic for the region from the MASAR experiment.

Keywords: Circulation-Slopewater-Modeling-Physical Oceanography

1 Circulation- A Short History

Ocean circulation can be described as either wind-driven or thermohaline. Thermohaline circulation develops as gradients of temperature and salinity form. Wind-driven circulation is due to the force of the wind on the surface of the water. Slopewater circulation is wind-driven. The forces acting on this type of circulation are Coriolis, pressure, and friction. The Coriolis acceleration is an apparent force due to the rotating earth. The pressure force is due to the local changes in density of the water column. Frictional forces develop due primarily to the wind stress which arises from the interaction of the wind with the surface of the water.

Many studies have been carried out on wind-driven circulation over the past fifty years. In particular, Sverdrup wrote *Wind-Driven Currents in a Baroclinic Ocean: with Application to the Equatorial Currents of the Eastern Pacific* in 1947. He assumes the following conditions:

- the pressure gradient is balanced solely by the Coriolis force,
- horizontal velocities and pressure gradient disappear with depth,
- lateral stresses are neglected,
- and the flow is steady-state.

Stommel followed up Sverdrup's work with his publication of *The Westward Intensification of Wind-Driven Ocean Currents* in 1948. In order to counterbalance any accelerations that could develop within the realm of the model, Stommel added frictional forces in the forms of wind stress and bottom friction. These assumptions allowed him to develop a general solution to ocean gyres and western currents. Two years later in 1950, Munk wrote *On the wind-driven ocean circulation*. In his research, Munk combines the work of Sverdrup and Stommel to produce a more accurate representation of ocean circulation. He assumes a baroclinic ocean (the pressure and density surfaces are at a slight angle to each other rather than parallel), lateral friction as the dissipative force, a basin that encompasses both northern and southern hemispheres, and uses the observed wind instead of a sinusoidal function.

Small contributions were made to the understanding of the Gulf Stream and slope-water region over the next thirty years. The next large contribution was added more recently. In the mid-80's, the Mid-Atlantic Continental Slope and Ocean Rise (MASAR) experiment was carried out. This program was designed to meet the objectives of the Mineral Management Service (a U.S. government department) which were

- to determine the general circulation features on the continental slope and rise
- to define and quantify the variability of winds, currents, transports, and other physical properties in these areas
- to determine the effect of the slope/rise circulation features on the physical oceanography of the Middle Atlantic Continental Shelf.

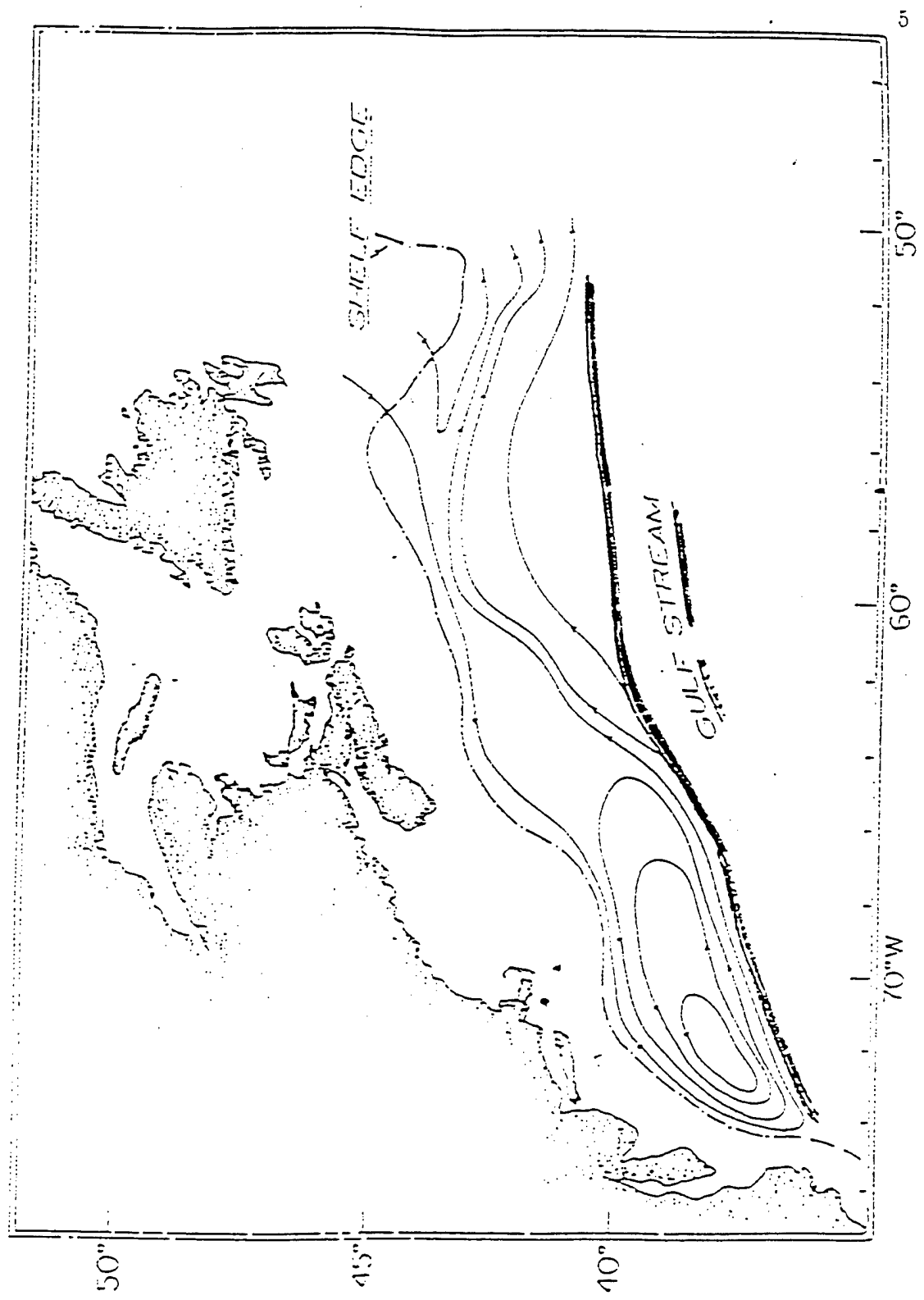


Figure 1: Schematic of Observational Data in the Slopewater Region from MASAR

The MASAR experiment involved several cruises in the region to put into place many data collection tools such as current meters and to collect water samples and real time data. Csanady was the key person for the interpretation of data for upper slope circulation in the MASAR experiment. He compiled all the major sources of information on the slopewater region and produced a simple, linear dynamical model of circulation in the area. His model is developed under the following assumptions:

- the flow is steady-state,
- Coriolis is balanced by wind stress,
- and the region is rectangular.

The model also includes non-zero boundary conditions for the streamfunction. Csanady and Hamilton's paper *Circulation of Slopewater* (1988) included an observational schematic (Figure 1) from the MASAR experiment showing the details of Slopewater circulation as it thought to occur. This paper is the springboard for the numerical scheme that was developed to reproduce this schematic.

2 The Model

The development of the model began with the following basic assumptions:

- the water parcel is a homogeneous layer of incompressible fluid,
- the region is enclosed in a rectangular box,
- there are zero boundary conditions,
- and the flow is steady-state.

The density is assumed to be constant throughout the parcel because it is a relatively shallow layer of approximately 500 meters. Therefore it is reasonable to assume incompressibility. The formulations that are the foundation of modeling slopewater circulation are presented in sections 2.1 through 2.4 of this paper.

2.1 Compressibility

The compressibility of a liquid β is defined by

$$\beta = -\frac{1}{V} \frac{dV}{dp} = -\left(\frac{1}{V} \frac{dV}{dt}\right) \left(\frac{dt}{dp}\right) = -\left(\frac{1}{V} \frac{dV}{dt}\right) \div \left(\frac{dp}{dt}\right) \quad (1)$$

where β is the compressibility, V is the volume, p is the pressure, $\frac{dV}{dt}$ is the change of volume with time, and $\frac{dp}{dt}$ is the change of pressure with time.

If the fluid is incompressible then β must be zero. Therefore

$$\left(\frac{1}{V}\right) \left(\frac{dV}{dt}\right) = 0 \quad (2)$$

If the mass of fluid is constant over volume, V and density is $\rho = m/V$ then

$$\frac{1}{\rho} \frac{d\rho}{dt} = \frac{V}{m} \frac{d}{dt} \left(\frac{m}{V} \right) = -\frac{1}{V} \frac{dV}{dt} = 0 \quad (3)$$

Hence incompressibility occurs when $\frac{1}{V} \frac{dV}{dt} = 0$ or $\frac{1}{\rho} \frac{d\rho}{dt} = 0$ [Pond and Pickard, 1983]. Incompressibility is an important assumption in this problem for it allows the stream function, ψ , to exist.

2.2 Stream Function

The total material derivative of the fluid density is known to be

$$\frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \vec{v} \cdot \nabla\rho \quad (4)$$

For an incompressible fluid, the density does not change. In this case

$$\frac{D\rho}{Dt} = 0 \quad (5)$$

The equation of continuity which is given by

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}(\rho\vec{v}) = 0 \quad (6)$$

can be expanded in the form of

$$\frac{\partial\rho}{\partial t} + \vec{v} \cdot \nabla\rho + \rho \nabla \cdot \vec{v} = 0. \quad (7)$$

By substitution, the equation of continuity of an incompressible fluid becomes

$$\nabla \cdot \vec{v} = 0, \quad (8)$$

which is expressed below for a two-dimensional flow velocity, $\vec{v} = \langle u, v \rangle$ as

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (9)$$

For the equation of continuity to be satisfied, let $u = \frac{\partial\psi}{\partial y}$ and $v = -\frac{\partial\psi}{\partial x}$, where ψ is the two-dimensional streamfunction of fluid dynamics. The velocity field \vec{v} is now expressed in vector form

$$\vec{v} = \left\langle \frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x} \right\rangle \quad (10)$$

When ψ is a constant, the streamfunction produces curves known as streamlines. A streamline is a curve in space drawn so that the velocity vectors are tangent to the curve. This allows the velocity field to be represented by streamlines.

2.3 Stommel's Gulf Stream

Stommel's Gulf Stream model is the pattern for the slopewater model. His model is based on the following assumptions:

- the ocean is rectangular with the y-axis pointing northward and the x-axis eastward
- the boundaries of the rectangle are between $0 < x < \lambda$ and $0 < y < b$
- the ocean is a homogeneous layer of constant depth D when at rest
- when there are currents, the total depth is $D + h$ where the depth varies by h which is much smaller than D
- functional form of the wind stress acting over the area is $-F \cos(\frac{\pi y}{b})$
- the frictional dissipative forces are $-Ru$ and $-Rv$ where R is the coefficient of friction and u and v are the velocity components
- the Coriolis parameter f is introduced as a function of y .

The steady-state equations of motion are written in the following manner:
The equation for motion in the y direction is

$$0 = f(D + h)v - F \cos(\frac{\pi y}{b}) - Ru - g(D + h)\frac{\partial h}{\partial x} \quad (11)$$

The equation for motion in the x direction is

$$0 = -f(D + h)u - Rv - g(D + h)\frac{\partial h}{\partial y}. \quad (12)$$

These equations are cross-differentiated and the equation of continuity is applied to the equations of motion resulting in

$$v(D + h)\left(\frac{\partial f}{\partial y}\right) + \left(\frac{F\pi}{b}\right) \sin\left(\frac{\pi y}{b}\right) + R\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = 0 \quad (13)$$

To a first approximation, h is negligible compared to D . This allows the previous equation to be rewritten as

$$\alpha v + \gamma \sin\left(\frac{\pi y}{b}\right) + \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 \quad (14)$$

where $\alpha = \left(\frac{D}{R}\right) \left(\frac{\partial f}{\partial y}\right)$ and $\gamma = \frac{F\pi}{Rb}$.

The ocean is considered to be incompressible which allows the introduction of the stream function. From the previous equation, the general equation is stated as

$$\nabla^2 \psi + \alpha \frac{\partial \psi}{\partial x} = \gamma \sin\left(\frac{\pi y}{b}\right), \quad (15)$$

and assumed boundary conditions are

$$\psi(0, y) = \psi(\lambda, y) = \psi(x, 0) = \psi(x, b) = 0 \quad (16)$$

Stommel solves the nonhomogeneous differential equation by inspection and separation of variables for ψ (See Appendix B for an example of this method.). The resulting stream function is

$$\psi = \gamma \left(\frac{b}{\pi} \right)^2 \sin \left(\frac{\pi y}{b} \right) [pe^{Ax} + qe^{Bx} - 1], \quad (17)$$

where $p = \left(\frac{1-e^{B\lambda}}{e^{A\lambda}-e^{B\lambda}} \right)$ and $q = 1 - p$

In order to analyze the stream function numerically, Stommel introduced the following parameters: $\lambda = 10^9 \text{cm}$, $b = 2\pi \times 10^8 \text{cm}$, $D = 2 \times 10^4 \text{cm}$, $F = 1 \text{dyne/cm}^2$, and $R = 0.02$. F and R were picked arbitrarily in order to produce the correct physical features.

When Coriolis is considered to be a linear function of latitude, the streamlines indicate an intensification of current velocities along the western boundary. This is the region of the Gulf Stream [Stommel, 1948].

2.4 Csanady's Transport Stream Function

In Csanady and Hamilton (1988), a transport stream function for slopewater circulation was introduced. It is given by

$$\frac{\partial \psi}{\partial x} \frac{\partial f}{\partial y} - \frac{\partial \psi}{\partial y} \frac{\partial f}{\partial x} = W \quad (18)$$

where W is the value for the wind stress curl over the rectangular region. Here, ψ is the stream function. The approximations to the Coriolis acceleration are $\frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial x}$. The transport stream function is subject to the following boundary conditions:

$$\psi = \begin{cases} 0 & \text{if } x=0, 0 < y < b/2 \\ 0 & \text{if } y=0, 0 < x < a \\ -K \frac{2y-b}{b} & \text{if } x=0, b/2 < y < b \\ G \frac{y}{b} & \text{if } x=a, 0 < y < b \\ (G + K) \frac{x}{a} & \text{if } y=b, 0 < x < a \end{cases} \quad (19)$$

where G is the inflow from the Gulf Stream, K is the inflow of the Coastal Labrador Sea Water, x and y are the coordinates, a is the length of the box along the x -axis, and b is the length of the box along the y -axis.

The differential equation of Csanady and Hamilton was developed using several important assumptions that are mentioned below:

- the dimensions of the idealized basin are 200 km wide by 1600 km long
- the mean depth is 500 m overlying an inert, deep, very heavy water mass
- the Coriolis parameter is $f = 10^{-4} \text{s}^{-1}$ which increases northward at the rate of $\beta = 1.6 \times 10^{-11} \text{m}^{-1} \text{s}^{-1}$,

- the wind stress curl is one of the driving factors of the circulation,
- the flow is steady-state, and
- a rotated coordinate system is used where the cartesian coordinate system is rotated 67 deg from north.

By inspection, one can find the differences between the equation stated by Csanady and Hamilton and the equation given by Stommel in the previous section. Csanady and Hamilton discount any frictional dissipative force by discarding the Laplacian term. This helps to simplify the problem and the numerics associated with it. The boundary conditions are non-zero indicating an influence imposed upon the body by current flows. The wind stress curl is a constant independent of the y variable. (See [Csanady and Hamilton, 1988] for more details.)

3 Discretization

The first step in developing the numerical scheme is to discretize a differential equation which describes the flow. The equation selected was a compilation of Stommel's streamfunction and Csanady and Hamilton's stream function. The equation introduced in the previous section was modified to include a frictional dissipative force with the addition of the Laplacian. The differential equation that was the basis for all calculations is

$$\epsilon \nabla^2 \psi + \alpha \psi_x = W \quad (20)$$

where ϵ and α are arbitrary constants and W is the wind stress curl.

3.1 Rotation of Coordinates

The previous equation is oriented in the North-South coordinate system. To make the equation applicable to the slope sea region and for better comparison to Csanady and Hamilton's equation, it is necessary to rotate the coordinates by 67 deg from North. The rotation was accomplished in the following linear. In general:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \alpha & \sin \alpha \\ -\sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad (21)$$

where

$$x' = x \cos \alpha + y \sin \alpha, \quad (22)$$

and

$$y' = -x \sin \alpha + y \cos \alpha \quad (23)$$

are the rotated coordinates. In the slopewater case:

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos(2\pi - \alpha) & \sin(2\pi - \alpha) \\ -\sin(2\pi - \alpha) & \cos(2\pi - \alpha) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (24)$$

Using trigonometric identities, the previous equation becomes

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} \quad (25)$$

Here the rotated coordinates are given by

$$x' = x \cos \alpha - y \sin \alpha, \quad (26)$$

$$y' = x \sin \alpha + y \cos \alpha \quad (27)$$

By utilizing the previous two identities, it is possible to rotate Stommel's equation into Csanady and Hamilton's region for Slopewater. By performing this operation, it becomes apparent how the slopewater transport function was developed. The mathematical operations are outlined below:

$$\psi(x, y) = \psi'(x', y') \quad (28)$$

$$\psi_x(x, y) = \frac{\partial \psi'}{\partial x'} \frac{\partial x'}{\partial x} + \frac{\partial \psi'}{\partial y'} \frac{\partial y'}{\partial x} \quad (29)$$

$$\psi_x = \frac{\partial \psi'}{\partial x'} \cos \alpha + \frac{\partial \psi'}{\partial y'} \sin \alpha \quad (30)$$

$$\psi_{xx} = \frac{\partial}{\partial x} \left(\frac{\partial \psi'}{\partial x'} \cos \alpha + \frac{\partial \psi'}{\partial y'} \sin \alpha \right). \quad (31)$$

Let $\cos \alpha = c$ and $\sin \alpha = s$, then the second derivatives of the streamfunction become

$$\psi_{xx} = c^2 \frac{\partial^2 \psi'}{\partial x'^2} + 2cs \frac{\partial^2 \psi'}{\partial x' \partial y'} + s^2 \frac{\partial^2 \psi'}{\partial y'^2} \quad (32)$$

$$\psi_y = \frac{\partial \psi'}{\partial x'} \frac{\partial x'}{\partial y} + \frac{\partial \psi'}{\partial y'} \frac{\partial y'}{\partial y} \quad (33)$$

$$\psi_y = -\frac{\partial \psi'}{\partial x'} s + \frac{\partial \psi'}{\partial y'} c \quad (34)$$

$$\psi_{yy} = c^2 \frac{\partial^2 \psi'}{\partial y'^2} - 2cs \frac{\partial^2 \psi'}{\partial x' \partial y'} + s^2 \frac{\partial^2 \psi'}{\partial x'^2} \quad (35)$$

The Laplacian $\nabla^2 \psi$ is defined to be $\psi_{xx} + \psi_{yy}$. When this is performed on the equations for rotated coordinates, the sine and cosine terms drop out leaving the following equation:

$$\psi_{xx} + \psi_{yy} = \psi'_{x'x'} + \psi'_{y'y'} \quad (36)$$

From this point onward, all references to x, y , and ψ will be assumed to be in the rotated coordinate system unless stated otherwise.

The new equation in the rotated coordinate system becomes by substitution of the previous equation into Csanady and Hamilton's transport streamfunction

$$\varepsilon(\psi_{xx} + \psi_{yy}) + \alpha(c\psi_x + s\psi_y) = W \quad (37)$$

3.2 Finite Differences

Finite differencing is a numerical method used to describe a continuous region with discrete points. At each point there is an approximated value which describes the domain. A truncation error is produced for each approximation. This will be the difference between the exact solution and the numerical scheme. The method of approximation is based on the expansion of the Taylor series. For centered differences, the Taylor series is expanded both in the forward direction and in the backward direction, i.e. $u(x + \Delta x, y)$ and $u(x - \Delta x, y)$, subtracted, and divided by twice the incremental step (See [Ames, 1992] for specific details.)

For this problem the Laplacian of ψ and the first order derivatives of ψ must be approximated using the following finite centered differences (Appendix A):

$$\psi_{yy} \simeq \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2} \quad (38)$$

$$\psi_{xx} \simeq \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h^2} \quad (39)$$

$$\psi_y \simeq \frac{\psi_{i,j+1} - \psi_{i,j-1}}{2k} \quad (40)$$

$$\psi_x \simeq \frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} \quad (41)$$

where $h = \frac{\lambda}{n+1}$, $k = \frac{b}{m+1}$, and i and j reference the discrete points within the domain.

These approximations are substituted into the transport streamfunction equation with rotated coordinates. The finite differenced equation becomes

$$\varepsilon \left(\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2} + \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h^2} \right) + \alpha \left[c \left(\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} \right) + s \left(\frac{\psi_{i,j+1} - \psi_{i,j-1}}{2k} \right) \right] = W.$$

Solving for $\psi_{i,j}$ one is left with the approximation of the domain at each point.

$$\psi_{i,j} = \frac{W}{a1} - \frac{a2}{a1} \psi_{i,j+1} - \frac{a3}{a1} \psi_{i,j-1} - \frac{a4}{a1} \psi_{i+1,j} - \frac{a5}{a1} \psi_{i-1,j} \quad (42)$$

where $a1 = -2\varepsilon \left(\frac{1}{k^2} + \frac{1}{h^2} \right)$, $a2 = \frac{\varepsilon}{k^2} + \frac{\alpha s}{2k}$, $a3 = \frac{\varepsilon}{k^2} - \frac{\alpha s}{2k}$, $a4 = \frac{\varepsilon}{h^2} + \frac{\alpha c}{2h}$, and $a5 = \frac{\varepsilon}{h^2} - \frac{\alpha c}{2h}$.

3.3 Iterative Method

The iterative method used for the numerical scheme is the Gauss-Seidel method. This method owes its derivation from the linear algebra equation that is given by $A\vec{x} = \vec{b}$. This method is known as the method of successive displacements or iteration by single steps. Here, the calculations are based on the immediate use of the improved values (See [Ames, 1992] for more details.). The computation utilizes the following equation:

$$x_i^{(k)} = \frac{-\sum_{j=1}^{i-1}(a_{ij}x_j^{(k)}) - \sum_{j=i+1}^n(a_{ij}x_j^{(k-1)} + b_i)}{a_{ii}} \quad (43)$$

where k is the number of iterations and i and j correspond to the row and column entries in the matrix.

In the numerical scheme for slopewater circulation the number of iterations were determined by the numerical output. The program, written using the Mathematica package and run on a SPARC20 Sun station, was instructed to continue computation until the maximum error was on the order of 10^{-7} .

4 Calibration Techniques

Numerical schemes must be validated before one can feel confident that the output of the model is indeed reasonable. This was accomplished by two methods of calibration. The first was a comparison to the Stommel Gulf Stream model. The second was a boundary condition verification.

4.1 Slopewater vs. Gulf Stream

The first calibration was a comparison of numerical output to an exact solution. (See Figure 2.) The exact solution to Stommel's differential equation discussed previously was produced. This was then compared to the numerical output when the parameters of Stommel's model were introduced into the model. In addition the boundary conditions were set equal to zero and the forcing term was set equal to a sine function instead of a constant. The absolute value of the difference between the maximum values of the exact solution and the numerical solution was then calculated. The difference was on the order of 10^{-8} after 500 iterations.

4.2 Boundary Verification

The second calibration was a boundary condition verification. The stream function was determined to be $\psi(x, y) = \sin(\frac{\pi x}{\lambda}) \sinh(\frac{\pi y}{\lambda})$ by separation of variables. (See Appendices B and C.) The first normal mode was given as the northern boundary condition with all other boundaries being zero. In the numerical scheme, W , α , and s were set equal to zero while c was set equal to one. The numerical solution was once again compared to the exact solution (Figure 3).

5 Discussion of Model Outputs

Several runs of the model were performed using various parameters. The first set of parameters was based solely on the constants and boundary conditions given by Csanady and Hamilton. The second set of parameters was a combination of Csanady and Hamilton's parameters and self-imposed boundary conditions.

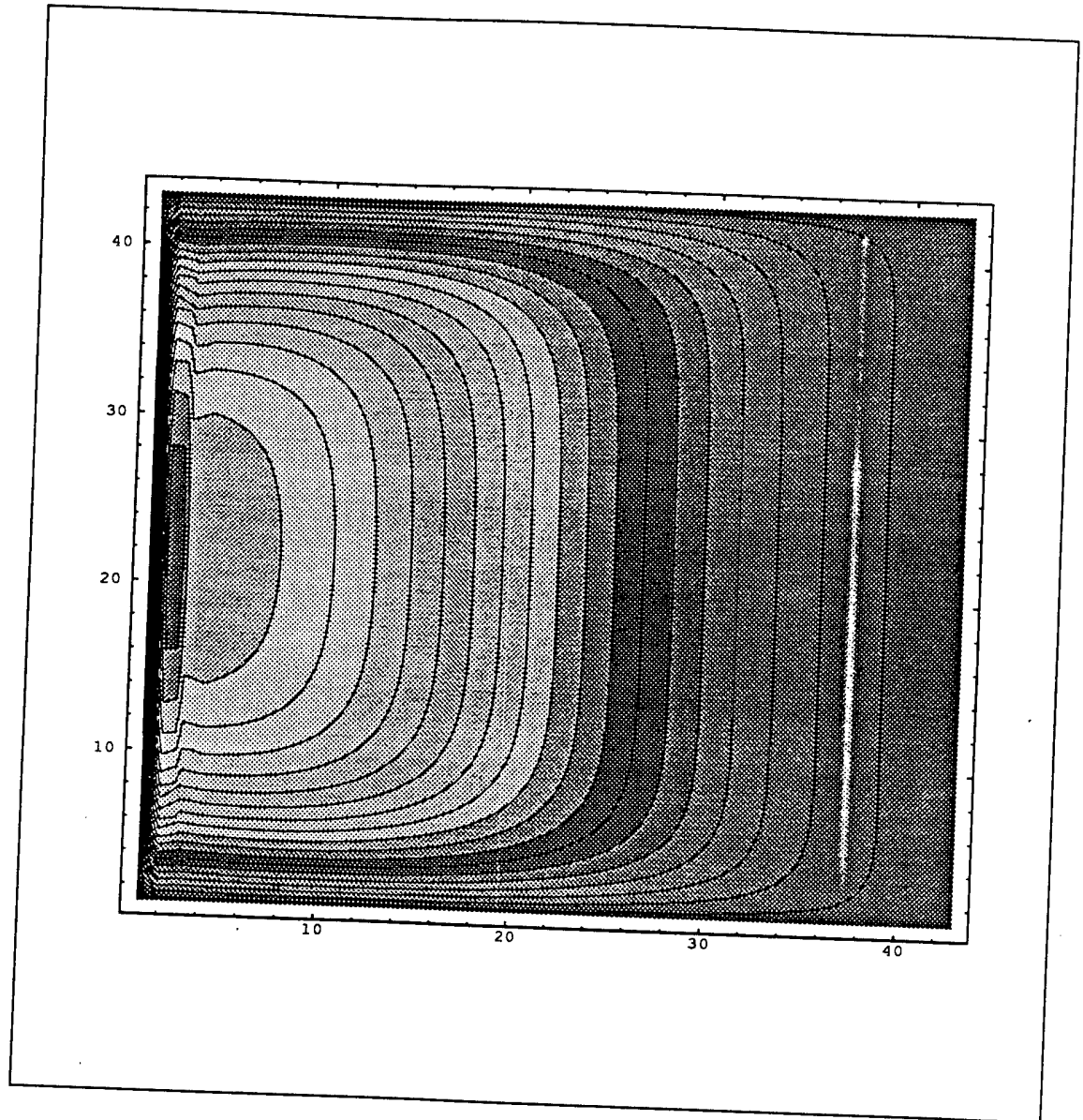


Figure 2: Stommel's model produced by the numerical model. (See Appendix D.2.) The lines represent the streamlines. The different shades represent different values of the streamfunction. The axes represent the dimensions of the calculated matrix.

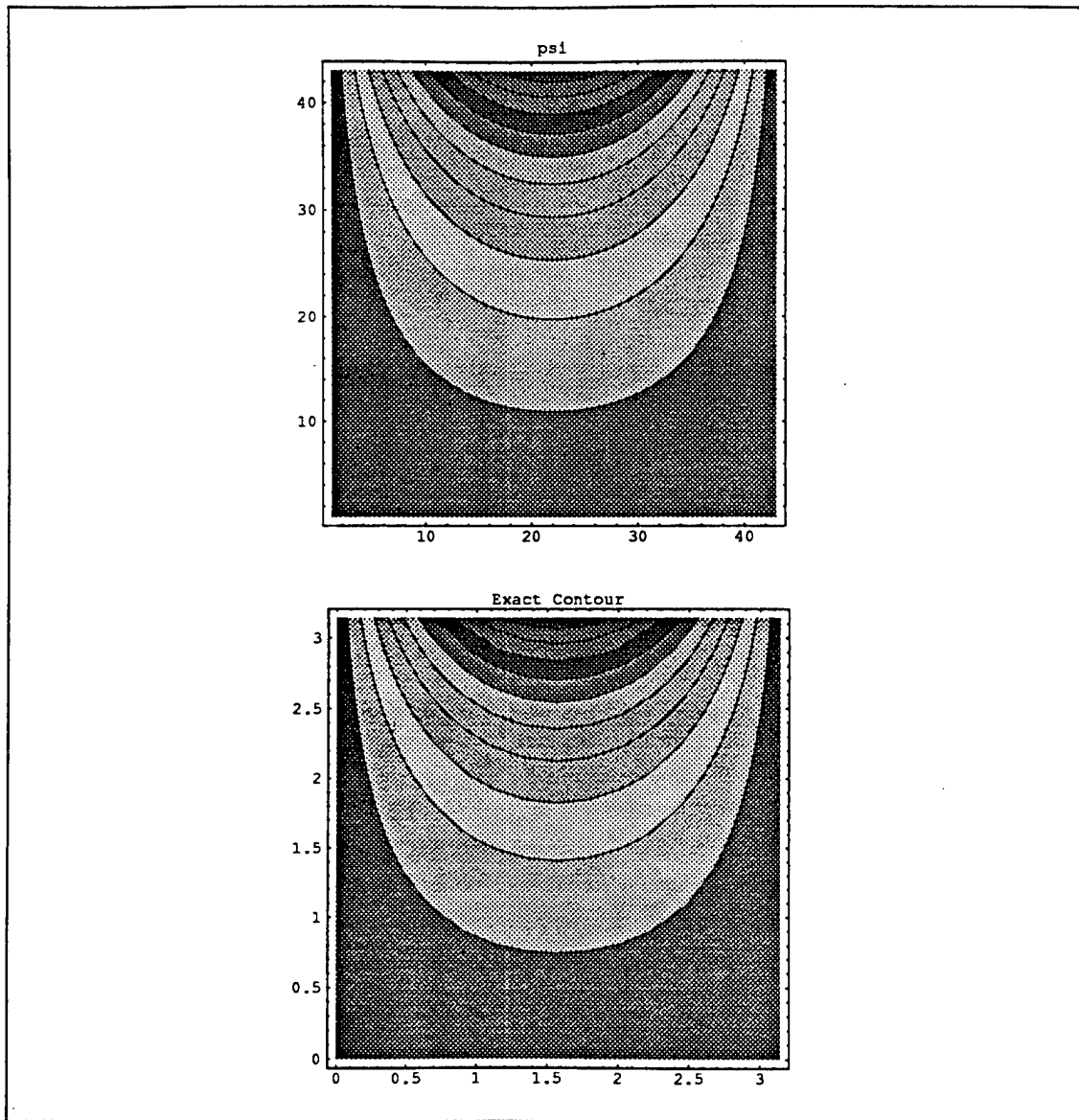


Figure 3: Boundary Verification by comparison of the exact solution of a specified streamfunction to the numerical solution of the same streamfunction. The curves are streamlines with different shades indicating different values of ψ . The exact contour has axes represented in units of length. The numerical solution has axes which represent the dimensions of the matrix.(See Appendix D.1 .)

5.1 Csanady and Hamilton's Parameters

Three different products were produced by varying the Gulf Stream constant, G , which is a value of flux rate of the Gulf Stream. In the case where $G = 2 \times 10^6 m^3 s^{-1}$, there is an intense crowding of stream lines near the northwestern boundary indicating a large velocity gradient. There is a small deflection eastward of the Gulf Stream. (See Figure 4.) As G increases to 6×10^6 and $10^7 m^3 s^{-1}$ there is an increased deflection of the Gulf Stream eastward and a decreasing velocity gradient in the northwest corner. (See Figures 5 and 6.). These products can be compared to the outputs of Csanady and Hamilton. A comparison shows that both models indicate greater deflection of the Gulf Stream with increased flux rate.

The boundary conditions imposed do not create a velocity field which corresponds to the field that is actually observed. (See Figure 1.) The velocity field only represents a small section of the slopewater region. In order to more accurately represent the circulation which is known to occur, new boundary conditions were developed and introduced into the model.

5.2 New Boundary Conditions

New boundary conditions were developed from inspection of the observational scheme. Based on the actions of the streamlines, functions were chosen to approximate the motion of the flow. The boundary condition on the continental shelf was determined to be $\psi(0, y) = 0$ because there is no fluid flow across the boundary, only parallel to the shelf. The boundary condition on the southwestern edge $\psi(x, 0) = 0$ because the flow changes directions but once again does not cross the boundary of the region. The final two boundaries were approximated by functions dependent on either x or y . The Gulf Stream side has two boundary conditions. For $\psi(1, y)$, when $0 < y < b/\lambda$, $\psi = 0$. When $b/\lambda < y < b/2\lambda$, ψ is approximated by a line. The Gulf Stream is represented in this fashion because after the current flows along the region for half of the distance, the stream enters into the slopewater. The northern edge must represent the change in direction of the fluid flow due to the exit of the Gulf Stream and the entrance of the Coastal Labrador Sea Water. This physical process is represented by a quadratic equation. (See Appendix C.)

The first run of the new boundary conditions (See Figure 7.) showed the development of a western gyre and the entrance of the Gulf Stream into the region. Because the output did not closely resemble the schematic (See Figure 1.), constants then had to be varied until the combination was discovered which would more accurately represent the schematic (Figure 7).

6 Data Analysis

To test the credibility of the numerical calculations, model outputs must be compared to real numbers. Data for comparison comes directly from the MASAR experiments. Charts of the region showing current meter deployment contain velocity vectors for the upper level currents. To retrieve the data, the resultant speed was determined by

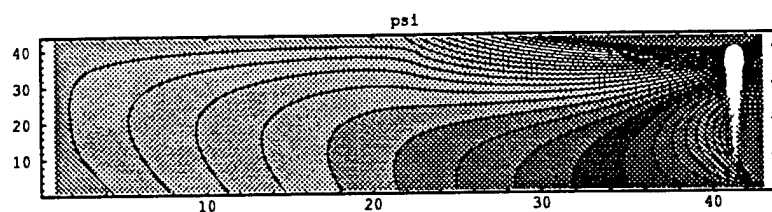


Figure 4: Slopewater model output (See Appendix D.4.) when using Csanady and Hamilton's parameters and boundary conditions. In this case, G is $2 \times 10^6 m^3 s^{-1}$. The curves are streamlines. The orientation of the output is: the northern edge is to the right of the page and the Gulf Stream is to the bottom. This orientation is the same for the next two products. The axes indicate the dimensions of the calculated matrix.

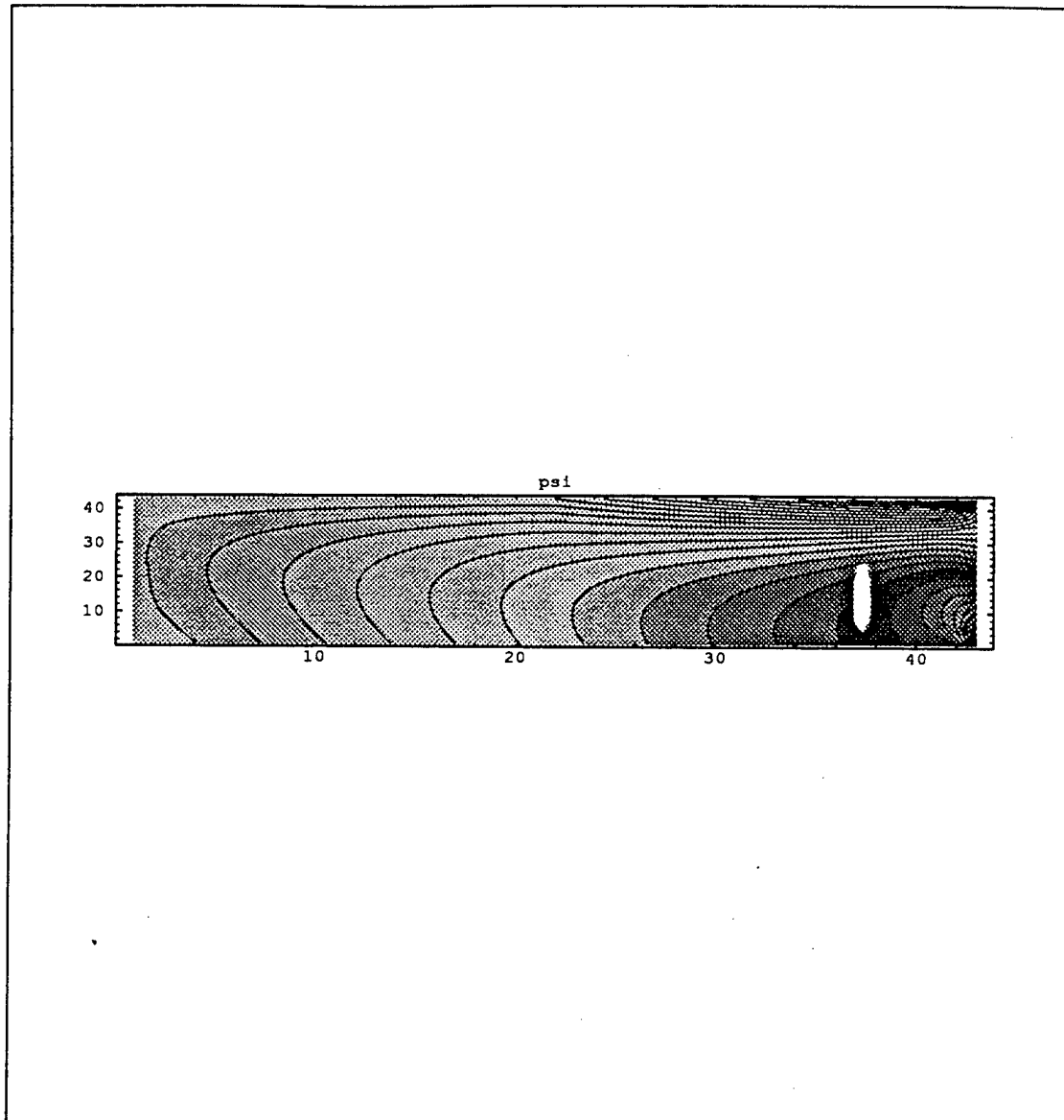


Figure 5: Slopewater output when using Csanady and Hamilton's parameters and boundary conditions. In this case, G is $6 \times 10^6 m^3 s^{-1}$ with the curves indicating the streamlines. The axes indicate the dimensions of the calculated matrix.

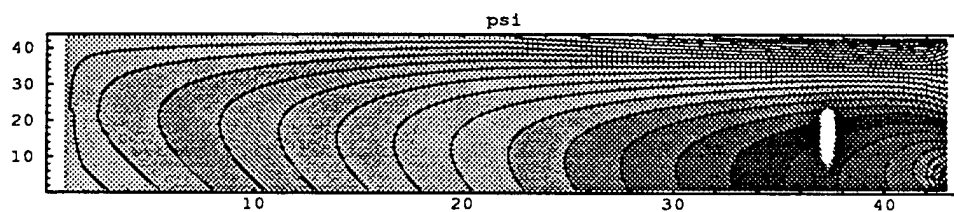


Figure 6: Slopewater output when using Csanady and Hamilton's parameters and boundary conditions. In this case, G is $10^7 m^3 s^{-1}$. The level curves are streamlines. The axes indicate the dimensions of the calculated matrix.

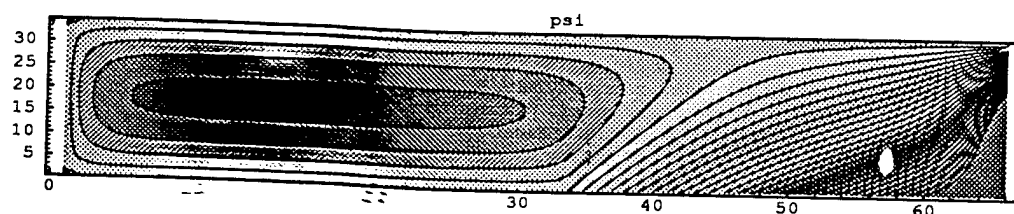


Figure 7: Slopewater circulation with new boundary conditions imposed. The level curves are streamlines. The axes show the number of nodes used in the matrix. The orientation of the output is the northern edge is to the right of the page and the Gulf Stream is to the bottom of the page. Some important features in the output are the closed gyre in the southern portion of the slope region and the intrusion of the Gulf Stream into the region. When compared to figure 1, one notices that the features are similar but the Coastal Boundary Current is overpowered by the Gulf Stream. (See Appendix D.4).

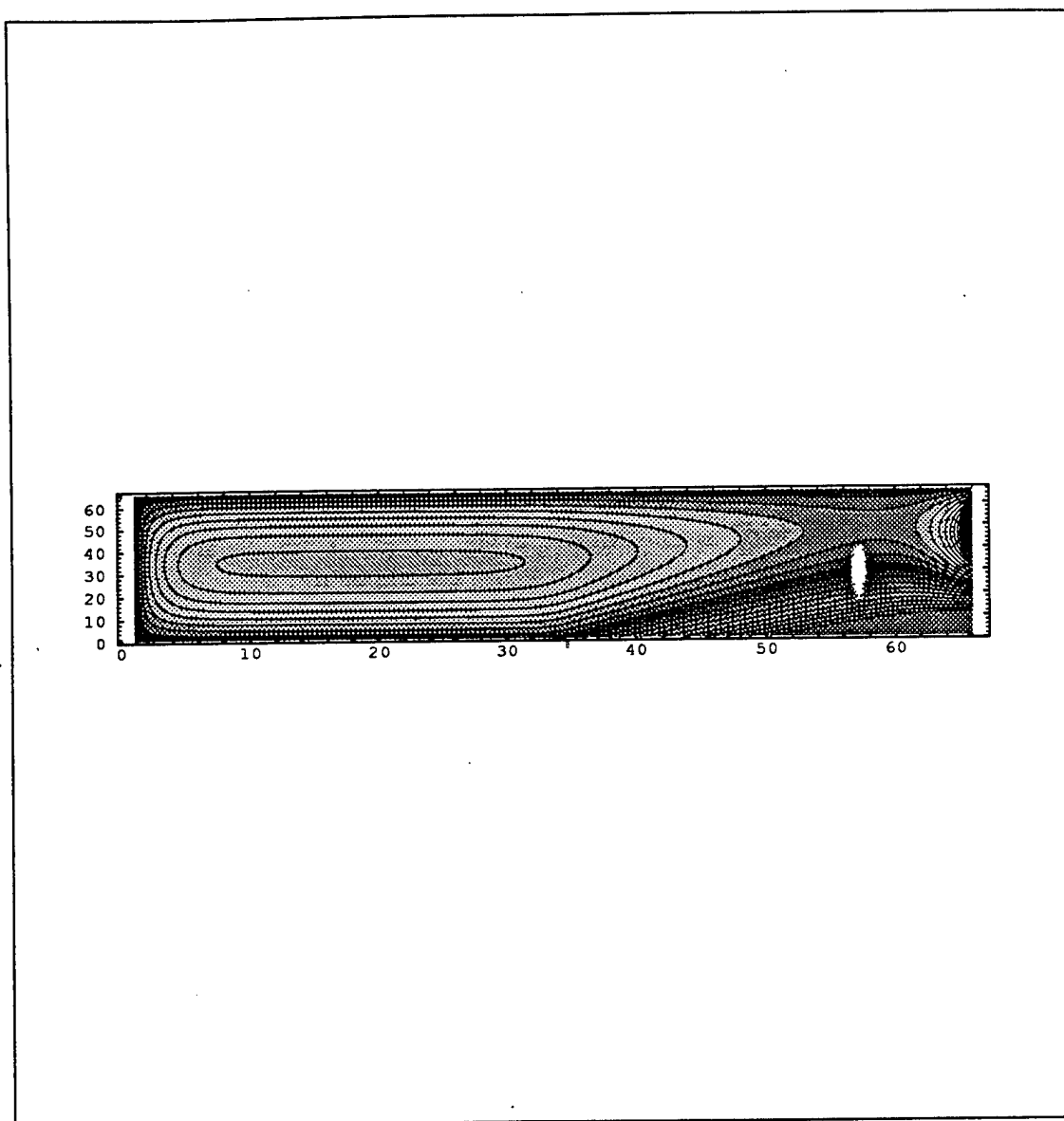


Figure 8: Slopewater circulation using new boundary conditions. This output is the result of many runs looking for the coefficients that would produce features that closely resemble Figure 1. Note the closed gyre, the intrusion of the Gulf Stream, the Coastal Labrador Current, and what appears to be the change in direction of the Coastal Labrador Current. The axes indicate the number of nodes. The level curves are streamlines. The orientation of the box is as follows: the northern edge is to the right of the page and the Gulf Stream is to the bottom of the page.

measuring velocities on a chart within the MASAR experiment report using the given scale. In order to compare the velocity vectors to the stream function, the resultant vectors had to be broken down into the u (the velocity in the x direction) and v (the velocity in the y direction) components. The stream function must be differentiated to derive the numerically calculated u and v components.

7 Naval Applications

The understanding of the coastal environment is important for military operations in the littoral environment. The dynamics within the area of operation affect many different warfares. In particular, knowledge of currents can be beneficial for successful mine operations, submarine warfare and anti-submarine warfare. Currents can even aid or hinder navigation.

The slopewater region is potentially a very important tactical region. The entire eastern seaboard is bounded by the water mass. The Gulf Stream has a large influence in the region and warm core eddies are typically spun off in this area. These eddies and the fast moving current of the Gulf Stream, if not fully understood, could hinder navigation and mine sweeping. The change in the temperature profile alters the sound velocity profiles when warm core eddies are interacting with the slopewater mass. This provides submarines a better chance to remain undetected. Knowledge and understanding of the circulation of the slopewater and how the eddies move within the circulation should give the tactical commander a better idea of where to search for the enemy or where to hide. The currents in the slopewater are not as fast along the shelf as they are along the Gulf stream region. Understanding this and being able to determine the speed changes as you move shoreward will benefit navigation.

The modeling of slopewater is applicable to many gyres throughout the world's oceans. The same techniques can be used to generate numerical schemes for any type of wind-driven circulation. If the Navy is interested in better understanding the coastal environments, numerical modeling is another approach that can be utilized.

8 Future Developments

The numerical scheme developed is highly simplified. It is two-dimensional, time independent, and based on a limited number of physical parameters. The region is rectangular and considered to be homogeneous and to have a constant depth. The wind stress curl is constant and the flux rate of the Gulf Stream is constant. The dynamic implications of warm core eddies and changes in temperature, pressure, and salinity are ignored.

In order to improve the model, each of these factors should be addressed. It is not a reasonable expectation to have every influence play an accurate role in the modeling of the region. At the present time, the knowledge and ability is unavailable to tackle a problem of that magnitude. The parameters must be addressed either one or two at a time. For example, the region could have non-rectangular dimensions developed and the wind stress curl could be made into a function instead of a constant. There are

many avenues of approach depending on the parameters of interest.

Other improvements could be made solely on the mathematical techniques of computation. Speed of computation and truncation error can be improved in a number of ways. The discretization method could be altered to produce a non-uniform grid which allows for better mapping of the region of interest. The iterative technique could be changed as well in order to increase speed of computation. There are several avenues of approach to numerical improvements which develop in higher level mathematics.

The numerical scheme has considerable room for improvement. However, the work accomplished is not trivial. The product resembles what is generally understood to be true about the circulation in the region. Improvements would allow for more accuracy and variation in the model outputs.

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A Development of Finite Difference Equations

$$\nabla^2 \psi|_{i,j} = \psi_{xx}|_{i,j} + \psi_{yy}|_{i,j}$$

$$\psi(x+h, y) = \psi + h\psi_x + \frac{h^2}{2!}\psi_{xx} + \frac{h^3}{3!}\psi_{xxx} + \frac{h^4}{4!}\psi_{xxxx}$$

$$\psi(x-h, y) = \psi - h\psi_x + \frac{h^2}{2!}\psi_{xx} - \frac{h^3}{3!}\psi_{xxx} + \frac{h^4}{4!}\psi_{xxxx}$$

$$\psi(x, y+k) = \psi + k\psi_y + \frac{k^2}{2!}\psi_{yy} + \frac{k^3}{3!}\psi_{yyy} + \frac{k^4}{4!}\psi_{yyyy}$$

$$\psi(x, y-k) = \psi - k\psi_y + \frac{k^2}{2!}\psi_{yy} - \frac{k^3}{3!}\psi_{yyy} + \frac{k^4}{4!}\psi_{yyyy}$$

Add the four equations together to get the following:

$$\psi_{i+1,j} + \psi_{i-1,j} + \psi_{i,j+1} + \psi_{i,j-1} = 4\psi_{i,j} + h^2\psi_{xx} + k^2\psi_{yy}$$

Therefore,

$$\psi_{xx} \sim \frac{1}{h^2}(\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j})$$

$$\psi_{yy} \sim \frac{1}{k^2}(\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1})$$

Given

$$\psi_x \sim \frac{1}{2h}(\psi_{i+1,j} - \psi_{i-1,j})$$

$$\psi_y \sim \frac{1}{2k}(\psi_{i,j+1} - \psi_{i,j-1})$$

$$h = \frac{\lambda}{n+1}$$

$$k = \frac{b}{m+1}$$

Substitute into the PDE for slopewater circulation.

$$\begin{aligned} \varepsilon(\psi_{xx} + \psi_{yy}) + \alpha(c\psi_x + s\psi_y) &= W \\ \varepsilon \left(\frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{k^2} + \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{h^2} \right) + \\ \alpha \left[c \left(\frac{\psi_{i+1,j} - \psi_{i-1,j}}{2h} \right) + s \left(\frac{\psi_{i,j+1} - \psi_{i,j-1}}{2k} \right) \right] &= W \end{aligned} \quad (44)$$

$$a1\psi_{i,j} = W - a2\psi_{i,j+1} - a3\psi_{i,j-1} - a4\psi_{i+1,j} - a5\psi_{i-1,j}$$

where $a1 = -2\varepsilon \left(\frac{1}{k^2} + \frac{1}{h^2} \right)$

$$a2 = \frac{\varepsilon}{k^2} + \frac{\alpha s}{2k}$$

$$a3 = \frac{\varepsilon}{k^2} - \frac{\alpha s}{2k}$$

$$a4 = \frac{\varepsilon}{h^2} + \frac{\alpha c}{2h}$$

$$a5 = \frac{\varepsilon}{h^2} + \frac{\alpha c}{2h}$$

B Separation of Variables

$$\psi_{xx} + \psi_{yy} = 0$$

Boundary Conditions

$$\psi(0, y) = 0$$

$$\psi(x, 0) = 0$$

$$\psi(\lambda, y) = 0$$

$$\psi(x, b) = 0$$

Suppose

$$\psi = F(x)G(y)$$

Then

$$\psi_{xx} = F''G$$

$$\psi_{yy} = FG''$$

Substitute

$$F''G = -FG''$$

Divide by $F''G''$

$$\frac{G}{G''} = -\frac{F}{F''} = -\mu$$

$$G = -G''\mu$$

$$G + G''\mu = 0$$

let $\sqrt{-\mu} = k$

$$G(y) = c_1 \sinh ky + c_2 \cosh ky$$

$$-F = -\mu F''$$

$$\mu F'' - F = 0$$

$$F(x) = c_3 \sin kx + c_4 \cos kx$$

The general solution of ψ becomes

$$\psi(x, y) = (c_1 \sinh ky + c_2 \cosh ky)(c_3 \sin kx + c_4 \cos kx)$$

To solve for the exact solution, boundary conditions must be applied.

$$\psi(0, y) = 0$$

$$0 = (c_1 \sinh ky + c_2 \cosh ky)(c_3 \sin 0 + c_4 \cos 0)$$

$$0 = c_4(c_1 \sinh ky + c_2 \cosh k)$$

$$c_4 = 0$$

$$\psi(x, 0) = 0$$

$$0 = (c_1 \sinh 0 + c_2 \cosh 0)(c_3 \sin kx)$$

$$0 = (c_2)(c_3 \sin kx)$$

$$c_2 = 0$$

$$\psi(x, y) = (c_1 \sinh ky)(c_3 \sin kx)$$

$$\text{Let } c_1 c_3 = c = 1$$

$$\psi(\lambda, y) = 0$$

$$0 = \sinh ky \sin k\lambda$$

$$0 = \sin k\lambda$$

$$k\lambda = n\pi$$

$$k = \frac{n\pi}{\lambda}$$

Therefore the exact equation is

$$\psi(x, y) = \sin \frac{n\pi x}{\lambda} \sinh \frac{n\pi y}{\lambda}$$

The boundary condition for $\psi(x, b)$ is designated to be the first normal mode of the exact solution. It is given by

$$\psi(x, b) = \sin \frac{\pi x}{\lambda} \sinh \frac{\pi b}{\lambda}$$

C Development of Boundary Conditions

The Shelf boundary condition is $\psi(0, y) = 0$.

The Southern edge boundary condition is $\psi(x, 0) = 0$.

The Gulf Stream boundary condition is $\psi(1, y) = 0$ when $0 < y < \frac{b}{2\lambda}$.

Where $\frac{b}{2\lambda} < y < \frac{b}{\lambda}$, $\psi(1, y) = \frac{2\lambda\delta}{b}(y - \frac{b}{2\lambda})$

δ is a point through which the line must pass. This point represents the maximum flux of the Gulf Stream.

The Gulf Stream boundary originates from the equation of a line which passes through the points $(\frac{b}{2\lambda}, 0)$ and $(\frac{b}{\lambda}, \delta)$. The derivation is as follows:

$$y = mx + B$$

$$x = \frac{y - B}{m}$$

$$m = \frac{\delta}{\frac{b}{2\lambda}}$$

$$x = \frac{2\lambda\delta}{b}(y - \frac{b}{2\lambda})$$

The Northern edge boundary condition is $\psi(x, \frac{b}{\lambda}) = \left[\frac{\delta(\xi + \xi^2) - (\eta + \delta\xi^2)}{\xi + \xi^2} \right] x^2 + \left[\frac{\eta + \delta\xi^2}{\xi + \xi^2} \right] x$

This boundary condition is based on a quadratic equation which must pass through the points $(0, 0)$, $(\xi, -\eta)$, and $(1, \delta)$. The derivation is as follows:

$f(x) = ax^2 + bx + c$, where a, b , and c are constants to be determined below.

To solve for the constants a, b , and c , substitute the coordinates into the quadratic equation.

Solve for c .

at $(0, 0)$

$$0 = 0 + 0 + c$$

$$c = 0$$

at $(\xi, -\eta)$

$$-\eta = a\xi^2 + b\xi$$

at $(1, \delta)$

$$\delta = a + b$$

Therefore, $a = \delta - b$.

Solve for b .

$-\eta = a\xi^2 + b\xi$ added to $-\delta\xi^2 = -a\xi^2 - b\xi^2$ yields

$$\eta + \delta\xi^2 = b(\xi + \xi^2)$$

$$b = \frac{\eta + \delta\xi^2}{\xi + \xi^2}$$

Solve for a by substitution.

$$a = \frac{\delta(\xi + \xi^2) - (\eta + \delta\xi^2)}{\xi + \xi^2}$$

So,

$$\psi(x, \frac{b}{\lambda}) = \left[\frac{\delta(\xi + \xi^2) - (\eta + \delta\xi^2)}{\xi + \xi^2} \right] x^2 + \left[\frac{\eta + \delta\xi^2}{\xi + \xi^2} \right] x$$

D Mathematica Programs

D.1 Boundary Verification

```
(* dimensions of box *)

lambda=10-9;

b=2*Pi*10-8;

(*depth of layer*)

d=2*10-4;

(*force of wind*)

F = 1;

(*frictional dissipative term*)

R=0.02;

(* constants *)

gamma=F*Pi/R/b;

eta=N[Pi/b];

coriolis=10(-13);

(* dimensions of matrix *)

m= 64;

n= 64;

(* incremental step size *)

k=b/lambda/(m+1);

h=1/(n+1);

disx=Table[N[i*h],{i,1,n}];

disy=Table[N[j*k],{j,1,m}];
```

```

(* constants *)

eps= 1;

alpha= 0;

(* cosine *)

c=0;

(* sine *)

s=0;

(* wind stress curl *)

W=Table[0,{i,m+2}];

(* constants *)

a1= N[-2*eps*(1/(k^2)+1/(h^2))];
a2= N[eps/(k^2)+alpha*s/(2*k)];
a3= N[eps/(k^2)-alpha*s/(2*k)];
a4= N[eps/(h^2)+ alpha*c/(2*h)];
a5= N[eps/(h^2)- alpha*c/(2*h)];

a12 = a2/a1; a13=a3/a1; a14=a4/a1; a15=a5/a1;

(* Boundary Conditions *)

g1[y_] = 0; f1[x_] = 0; g2[y_] = 0;
f2[x_] = N[Sin[Pi x]*Sinh[Pi]];

psi=Table[0,{i,m+2},{j,n+2}];

newpsi = psi;

Do[psi[[1,j+1]]=g1[disy[[j]]], {j, m}];

```

```

Do[psi[[n+2,j+1]]= g2[disy[[j]]], {j, m}];

Do[psi[[i+1,1]]= f1[disx[[i]]], {i, n}];

Do[psi[[i+1,m+2]]= f2[disx[[i]]], {i, n}];

Do[
  Do[newpsi[[i,j]]= W[[i-1]]/a1-a12*psi[[i,j+1]]-a13*psi[[i,j-1]]
    -a14*psi[[i+1,j]]-a15*psi[[i-1,j]], {i,2,n+1}, {j,2,m+1}];
  error=Max[Abs[psi-newpsi]]; Print[error];
  Do[psi[[i, j]]=newpsi[[i,j]], {i, 2, n+1}, {j, 2, m+1}],
{ii, 50}]

AA=ListContourPlot[Transpose[psi],ColorFunction->Hue]

exactsoln[x_,y_]=Sin[Pi x]*Sinh[Pi y*lambda/b]

zz=Table[exactsoln[disx[[i]],disy[[j]]],{i,n+2},{j,m+2}]

ZZ=ContourPlot[exactsoln[x,y],{x,0,1},{y,0,b/lambda},
  ColorFunction-> Hue]

AAA=Show[AA,PlotLabel->"numerical"]

PSPrint[AAA]

zzz=Show[ZZ,PlotLabel->" exact"]

PSPrint[zzz]

Max[Abs[zz-newpsi]]

AAAA= Show[GraphicsArray[{{AAA},{zzz}}]]

Display["graph1.ps", AAAA]

PSPrint[AAAA]

```

D.2 Stommel Verification

(* dimensions of box *)

$\lambda = 2 \times 10^{-7}$;

$b = 16 \times 10^{-7}$;

(* depth of layer *)

$d = 2 \times 10^{-4}$;

(* force of wind *)

$F = 1$;

(* frictional dissipative term *)

$R = 0.02$;

(* constants *)

$\gamma = F \cdot \pi / R / b \cdot \lambda$;

$\eta = N[\pi / b]$;

$\text{coriolis} = 10^{-13}$;

(* Gulf Stream flux rate *)

$G = 2 \times 10^{12}$;

(* Coastal Labrador Sea Water flux rate *)

$K = 4 \times 10^{12}$;

(* matrix dimensions *)

$m = 64$;

$n = 64$;

(* incremental step size *)

```

k=b/lambda/(m+1);

h=1/(n+1);

disy=Table[N[i*k],{i,1,m}];

disx=Table[N[-j*h],{j,1,n}];

(* constants *)

eps= 1;

alpha= 1*lambda;

(* cosine *)

c=0;

(* sine *)

s= 0;

(* wind stress curl *)

W=Table[1,{i,1,m}];

(* constants *)

a1= N[-2*eps*(1/(k^2)+1/(h^2))];

a2= N[eps/(k^2)+alpha*s/(2*k)];

a3= N[eps/(k^2)-alpha*s/(2*k)];

a4= N[eps/(h^2)+ alpha*c/(2*h)];

a5= N[eps/(h^2)- alpha*c/(2*h)];

a12 = a2/a1; a13=a3/a1; a14=a4/a1; a15=a5/a1;

(* Boundary Conditions *)

g1[x_]=0;g2[x_]=0;f1[y_]=0;f2[y_]=0;

```

```

psi=Table[0,{i,m+2},{j,n+2}];

newpsi = psi;

Do[psi[[1,j+1]]=g1[disx[[j]]], {j, n}];

Do[psi[[m+2,j+1]]= g2[disx[[j]]], {j, n}];

Do[psi[[i+1,1]]= f1[disy[[i]]], {i, m}];

Do[psi[[i+1,n+2]]= f2[disy[[i]]], {i, m}];

Do[
  Do[newpsi[[i,j]]= W[[i-1]]/a1-a12*psi[[i,j+1]]-a13*psi[[i,j-1]]
    -a14*psi[[i+1,j]]-a15*psi[[i-1,j]], {i,2,m+1}, {j,2,n+1}];
  error=Max[Abs[psi-newpsi]]; Print[error];
  Do[psi[[i, j]]=newpsi[[i,j]], {i, 2, m+1}, {j, 2, n+1}],
{ii, 50}]

AA=ListContourPlot[psi, PlotLabel->Stommel Verification,
  ColorFunction->Hue,Contours->20]

PSPrint[AA]

Display["stommolver.ps", AA]

```

D.3 Csanady's Boundaries

```

(* Gauss- Seidel, Stommel *)

(* The real psi is related to this by psi by
   lambda* this psi(x/lambda, y/lambda)*)

(* dimensions of box in meters *)

lambda=2*105;

b=16*105;scale=lambda/b;

(* depth *)

d=5*10-2;

(* force of wind *)

F=2.4*10-10;

(* frictional dissipative term *)

R=0.0002;

(* constants *)

gamma=F/R;gammabar=gamma*lambda;

(* beta plane approximation for Coriolis *)

beta= 1.6*10-11;

(* constants *)

alpha= d*beta/R; alphabar=alpha*lambda;

(* incremental step size *)

k=b/lambda/(m+1);

h=1/(n+1);

(* size of matrix *)

```



```

m= 41;

n= 41;

disx=Table[N[i*h],{i,1,n}];

disy=Table[N[j*k],{j,1,m}];

(* constants *)

eps= 1;

c= N[Cos[1.168]];

s= N[Sin[1.168]];

(* Gulf Stream flux rate *)

G= 2*10-6 (*m3/s*);

(* Coastal Labrador Current flux rate *)

K= 4*10-6 (*m3/s*);

(* wind stress curl *)

W=Table[gammabar,{j,1,m}] (*m/s*);

(* constants *)

a1= N[-2*eps*(1/(k2)+1/(h2))];

a2= N[eps/(k2)+alphabar*s/(2*k)];

a3= N[eps/(k2)-alphabar*s/(2*k)];

a4= N[eps/(h2)+ alphabar*c/(2*h)];

a5= N[eps/(h2)- alphabar*c/(2*h)];

a12 = a2/a1; a13=a3/a1; a14=a4/a1; a15=a5/a1;

(* Boundary Conditions *)

```

```

g2[y_] = G*y/b;
g1[y_] = If[0 < y < b/2/lambda, 0, -K*(2*y*lambda-b)/b/lambda];
f1[x_] = 0;
f2[x_] = (G+K)*x/lambda;
psi=Table[0.11,{i,n+2},{j,m+2}];
Do[psi[[1,j+1]]=g1[disy[[j]]], {j, m}];
Do[psi[[n+2,j+1]]= g2[disy[[j]]], {j, m}];
Do[psi[[i+1,1]]= f1[disx[[i]]], {i, n}];
Do[psi[[i+1,m+2]]= f2[disx[[i]]], {i, n}];
oldpsi = psi; error =1; ii=1;
newdisx=Append[Prepend[disx, 0], N[1]];
newdisy=Append[Prepend[disy, 0], N[1/scale]];
Print["m = ", m, "      n = ", n];
While[error > 10^-7,
  Do[psi[[i,j]]=W[[j-1]]/a1-a12*psi[[i,j+1]]-a13*psi[[i,j-1]]
    -a14*psi[[i+1,j]]-a15*psi[[i-1,j]], {i,2,n+1}, {j,2,m+1}];
  error=Max[Abs[psi-oldpsi]];Print[ii," ", error];
  Do[oldpsi[[i, j]]=psi[[i,j]], {i, 2, n+1}, {j, 2, m+1}];
  ii=ii+1];

AA1=ListContourPlot[Transpose[psi], PlotLabel->"psi", Contours->20,
  ColorFunction-> Hue, AspectRatio->3,ColorOutput->CMYKColor]

PSPrint[AA1]

Display["graph3.ps", AA1]

```

D.4 New Boundaries

```

(* Gauss- Seidel, Stommel *)

(* The real psi is related to this by psi by
   lambda* this psi(x/lambda, y/lambda)*)

(* dimensions of box *)

lambda=1;

b=10 ;scale=lambda/b;

(* depth *)

d=1;

(* force of wind *)

F=1;

(* frictional dissipative term *)

R=1;

(* constants *)

gamma=F/R;gammabar=gamma*lambda;

beta= 1;

alpha= d*beta/R; alphabar=alpha*lambda;

(* incremental step size *)

k=b/lambda/(m+1);

h=1/(n+1);

(* size of matrix *)

m= 64;

n= 32;

```

```

disx=Table[N[i*h],{i,1,n}];
disy=Table[N[j*k],{j,1,m}];

(* constants *)

eps= 1;

c= N[Cos[1.168]];

s= N[Sin[1.168]];

(* Gulf Stream flux rate *)

G= 1 (*m^3/s*);

(* Coastal Labrador current flux rate *)

K= 1 (*m^3/s*);

(* constants *)

mu=0.25;

eta=1;

(* wind stress curl *)

W=Table[gamma_bar,{j,1,m}] (*m/s*);

(* constants *)

a1= N[-2*eps*(1/(k^2)+1/(h^2))];
a2= N[eps/(k^2)+alpha_bar*s/(2*k)];
a3= N[eps/(k^2)-alpha_bar*s/(2*k)];
a4= N[eps/(h^2)+ alpha_bar*c/(2*h)];
a5= N[eps/(h^2)- alpha_bar*c/(2*h)];

a12 = a2/a1; a13=a3/a1; a14=a4/a1; a15=a5/a1;

```

```

(* Boundary Conditions *)

gulf[y_] = If[0<y<b/2/lambda,0,2*G*lambda/b*(y-b/2/lambda)];

shelf[y_] = 0;

south[x_] = 0;

north[x_] = ((G*(mu+mu^2)-(eta+G*(mu^2)))/(mu+mu^2))*x^2
            +(eta+G*(mu^2))/(mu+mu^2)*x;

psi=Table[0.11,{i,n+2},{j,m+2}];

Do[psi[[1,j+1]]=shelf[disy[[j]]], {j, m}];

Do[psi[[n+2,j+1]]= gulf[disy[[j]]], {j, m}];

Do[psi[[i+1,1]]=south[disx[[i]]], {i, n}];

Do[psi[[i+1,m+2]]= north[disx[[i]]], {i, n}];

oldpsi = psi; error =1; ii=1;

newdisx=Append[Prepend[disx, 0], N[1]];

newdisy=Append[Prepend[disy, 0], N[1/scale]];

Print["m = ", m, " n = ", n];

While[error > 10^-7,
  Do[psi[[i,j]]=W[[j-1]]/a1-a12*psi[[i,j+1]]-a13*psi[[i,j-1]]-
    a14*psi[[i+1,j]]-a15*psi[[i-1,j]], {i,2,n+1}, {j,2,m+1}];
  error=Max[Abs[psi-oldpsi]];Print[ii," ", error];
  Do[oldpsi[[i, j]]=psi[[i,j]], {i, 2, n+1}, {j, 2, m+1}];
  ii=ii+1];

AA1=ListContourPlot[Transpose[psi], PlotLabel->"psi ",
  Contours-> 20, ColorFunction-> Hue,AspectRatio->3,
  ColorOutput->CMYKColor]

PSPrint[AA1]

Display["graph6.ps", AA1]

```